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(a, d) EDGE- ANTIMAGIC TOTAL LABELLING OF NON-PLANAR CLASSES OF GRAPHS

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ABSTRACT

The (a, d) edge- antimagic total labeling of a graph G is a bijective function ρ: V (G) ∪E(G) → {1, 2,…,p + q} such that the set of edge-weights of all edges in G, {w(xy) = ρ(x) + ρ(xy) + ρ(y) : xy ∈E(G)}, forms an arithmetic progression {a, a+ d, a+ 2d,..., a+ (q − 1)d}, where a > 0 and d ≥ 0 are fixed integers. The planar and non-planar classes of graphs have an important place for the computation of their (a, d) edge- antimagic total labelling apart from the many applications of such labeling in computer sciences and elsewhere. The present article covers the same type of labeling of non-planar families of graphs.

KEYWORDS: Edge-Antimagic Total Labelling, Planar Graphs & Non-Planar Graphs

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INTRODUCTION

The notations $V(G)$ and $E(G)$ denote the vertex set and the edge set in a simple graph G. A (p, q)-graph G is one with $|V(G)| = p$ and $|E(G)| = q$. Moreover, the theoretic ideas of graphs can be seen in [21]. A labeling (or valuation) of a graph is in fact a mapping that carries elements of the graph to numbers (usually to positive or non-negative integers). Here, the domain will be $V(G) \cup E(G)$. In other words, the labelling in this article is total labelling. In some labellings only the vertex set or the edge set will be used and we shall call them vertex-labelling or edge-labellings, respectively. Graph labellings has many types such as harmonius, radio, cordial, graceful and antimagic. The recent survey of graph labellings can be seen in [5]. In this paper, we will focus on antimagic total labelling type. In [1], more details on an antimagic total labeling can be seen. The notion of edge-magic total labelling of graphs derives its origin in the research work of A. Kotzig and A. Rosa [12, 13] for which they used the terminology magic valuation. Let us now move to few useful definitions and relevant research work previously done.

Definition 1

A (a, d) edge-antimagic vertex ((a, d)−EAV) labelling of a graph G is a bijection $\rho: V(G) \to \{1, 2, \cdots, p\}$ such that the set of edge-sums of all edges in G, $\{w(xy) = \rho(x) + \rho(y) : xy \in E(G)\}$, forms an arithmetic progression $\{a, a + d, a + 2d,..., a + (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are fixed integers.

R. Simanjuntak *et al.*, [19] proved that cycles and path, C_{2n+1}, P_{2n+1} and P_{2n}, have an (n + 2, 1)-EAV labeling when $n \geq 1$. They further proved that the odd path P_{2n+1} has a (n + 3, 1)-EAV labelling and the path P_n admits a (3, 2)-EAV labelling for $n \geq 1$. In [3], M. Baca *et al.*, proved that if a connected graph G (must not be a

tree) has an (a, d)-EAV labelling then $d = 1$. Further that a cycle C_n has no (a, d)-EA V labeling for $d > 1$ and $n \geq 3$ [3].

Definition 2

A (a, d) edge-antimagic total labeling of a graph G is a bijection $\rho: V(G) \cup E(G) \rightarrow \{1, 2, \cdots, p + q\}$ such that the set of edge-weights of all edges in G, $\{w(xy) = \rho(x) + \rho(xy) + \rho(y) : xy \in E(G)\}$, forms an arithmetic progression $\{a, a+ d, a+ 2d,..., a+ (q - 1)d\}$, where $a > 0$ and $d \geq 0$ are fixed integers. The graph G, if admits such labeling, is called an (a, d) edge-antimagic total graph. (abbreviated as (a, d) − EAT labeling/ graph)

Definition 3

A (a, d) -EAT labelling ρ is called a super (a, d) edge antimagic total labelling of G if $\rho(V(G)) \rightarrow \{1, 2, \cdots, v\}$. Thus, a super ($a$, d)- edge-antimagic total graph is a graph that admits a super (a, d) edge-antimagic total labelling. (abbreviated as super (a, d) − EAT labelling/ graph). If $d = 0$, then a s uper (a, 0)-EAT labelling is called a super edge-magic total labelling and a is called a magic constant or valence. For d other than 0, a is called minimum edge-weight.

The definition of an (a, d)- EAT labelling was established by R. Simanjuntak, Bertault and M. Miller in [19] as a natural extension of an edge-magic total labelling defined by A. Kotzig and A. Rosa earlier. A super (a, d)- EAT labeling is a further natural extension of the notion of a super (a, 0)-EAT labelling introduced by Hikoe Enomoto *et al.*, in [4]. And the following interesting conjecture of the same paper that every tree admits a super (a, 0) edge-antimagic total labelling. Many researchers have pillared this conjecture by deriving super (a, d)- EAT labeling for many particular classes of trees. As in, stars, path like trees, W-trees, subdivided stars, caterpillars and lobsters. Such results can be seen in [2, 6, 7, 9, 10, 11, 8, 15, 17, 18, 20].

MAIN RESULTS

This article pillars upon the calculation of (a, 2) and hence (a, 0) edge- antimagic total labelling of planar and non-planar classes of graphs. The relevant edge-vertex connection of our graphs shall be defined in the corresponding result. The following theorems are our main results.

This part of our main results deals with twisted ladder $TL_{l,m}$ and super edge-magicness of a graph related to twisted ladder. This graph is a symmetric generalization of the generalized ladder, which is our motivation here as well.

Definition 4

The twisted ladder, denoted by $T\,L_{l,m}$ is a graph of order $m(2l - 1)$ with following vertex and edge sets, where l is chosen to be odd.

$$V(TL_{l,m}) = \{t_i^j : 1 \leq i \leq m, 1 \leq j \leq l, j \equiv 1 (\mathrm{mod}\, 2)\}$$
$$\cup \{x_i^j, y_i^j, z_i^j : 1 \leq i \leq m, 2 \leq j \leq l-1, j \equiv 0 (\mathrm{mod}\, 2)\}.$$

$$E(TL_{l,m}) = \{x_i^j t_i^{j+1}, y_i^j t_i^{j+1}, z_i^j t_i^{j+1} : 1 \leq i \leq m, 2 \leq j \leq l-1, j \equiv 0 (\mathrm{mod}\, 2)\}$$
$$\cup \{t_i^j x_i^{j+1}, t_i^j y_i^{j+1}, t_i^j z_i^{j+1} : 1 \leq i \leq m, 1 \leq j \leq l-2, j \equiv 1 (\mathrm{mod}\, 2)\}$$
$$\cup \{x_i^j y_i^j : 1 \leq i \leq m, 2 \leq j \leq l-1, j \equiv 0 (\mathrm{mod}\, 2)\}$$
$$\cup \{t_i^j t_{i+1}^j : 1 \leq i \leq m-1, 1 \leq j \leq l, j \equiv 1 (\mathrm{mod}\, 2)\}.$$

Note that twisted ladder itself is a planar graph. In fact, we have found super edge-magic labelling of a graph

related to

$TL_{l,m}$. Let us denote this graph by H_1, where

$$V(H_1) = V(TL_{l,m})$$

$$E(H_1) = E(TL_{l,m}) \cup \{y_1^j t_2^{j+1} : 2 \le i \le l-1, j \equiv 0 (\text{mod } 2)\}.$$

Theorem 1

For $m \ge 3$ odd, the graph H_1 admits super $(\frac{m+5}{2}, 2)$ and $(\frac{m+5}{2}, 0)$ edge- antimagic total labellings.

Proof. We are defining a labelling $\delta : V(H_1) \to \{1,2,...,m(2l-1)\}$ as follows:

For $1 \le j \le l$, $j \equiv 1 (\text{mod } 2)$

$$\delta(t_i^j) = \begin{cases} \frac{1}{2}(4m(j-1)+i+1) : 1 \le i \le m; i \equiv 1(\text{mod } 2); \\ \\ \frac{1}{2}(4m(j-3)+i+1) : 2 \le i \le m-1; i \equiv 0(\text{mod } 2); \end{cases}$$

For $2 \le j \le l-1$, $j \equiv 0 (\text{mod } 2)$

$$\delta(x_i^j) = \begin{cases} \frac{1}{2}(m(4j-5)+i) : 1 \le i \le m; i \equiv 1(\text{mod } 2); \\ \\ \frac{1}{2}(2m(2j-3)+i) : 2 \le i \le m-1; i \equiv 0(\text{mod } 2); \end{cases}$$

$$\delta(y_i^j) = 2mj - i + 1 : 1 \le i \le m;$$

$$\delta(z_i^j) = m(2j-1) - i + 1 : 1 \le i \le m;$$

It can easily be observed that with suitable edge-labels, this labelling δ refers to an $(a, 2)$ and $(a, 0)$ edge-antimagic total labelling of H_1, where these labels are assigned in same and in opposite order respectively with $a = \frac{m+5}{2}$.

We are in a position to propose the following open problems for researchers.

Open Problem 1

For even m and for any value of a, can you determine a super $(a, 2)$ edge- antimagic total labelling of H_1?

Open Problem 2

Determine a super $(a, 2)$ edge- antimagic total labelling of twisted ladder $TL_{l,m}$ given in Definition 4.

This next part of our main results deals with the construction and super (a, 2) edge-antimagicness of another newly developed special family of graphs, which is non-planar as well as it contains cycles of all lengths from 3 to the order of the graph. We define here two non-isomorphic such non-planar families and denote them by G_1 and G_2 respectively.

Definition 5

For a path P_n, where n is even, G_1 is a non-planar graph of order $4n$ with following vertex-edge connection:

$$V(G_1)=\{x_i, y_i, z_i : 1\le i\le n\}\cup\{t_i, r_i : 1\le i\le \frac{n}{2}\}$$

$$\begin{aligned}E(G_1)&=\{x_i\,y_i, y_i\,z_i : 1\le i\le n\}\cup\{t_i\,x_{2i-1}, t_i\,x_{2i}, r_i\,z_{2i-1}, r_i\,z_{2i} : 1\le i\le \frac{n}{2}\}\\&\cup\{x_i\,x_{i+1}, y_i\,y_{i+1}, z_i\,z_{i+1} : 2\le i\le n-2, i\equiv 0(\mathrm{mod}\,2)\}\\&\cup\{x_i\,r_{\frac{i+1}{2}} : 1\le i\le n-1, i\equiv 1(\mathrm{mod}\,2)\}\cup\{t_i\,z_{2i}, t_i\,y_{2i-1} : 1\le i\le \frac{n}{2}\}\\&\cup\{x_i\,z_{i-1}, y_i\,r_{\frac{i}{2}} : 2\le i\le n, i\equiv 0(\mathrm{mod}\,2)\}\end{aligned}$$

Theorem 2

The non-planar graph G_1 defined as above admits a super (3, 2) and (3, 0) edge- antimagic total labelling.

Proof. We define a labelling $\alpha : V(G_1)\to\{1,2,...,4n\}$ as follows:

$$\alpha(x_i)=2(2i-1) : 1\le i\le n;$$

$$\alpha(y_i)=\begin{cases}4i-3 : 1\le i\le n-1; i\equiv 1(\mathrm{mod}\,2);\\ 4i : 2\le i\le n; i\equiv 0(\mathrm{mod}\,2);\end{cases}$$

$$\alpha(z_i)=4i-1 : 1\le i\le n;$$

$$\alpha(t_i)=4(2i-1) : 1\le i\le \frac{n}{2};$$

$$\alpha(r_i)=8i-3 : 1\le i\le \frac{n}{2};$$

With suitable edge- labels, this labelling α refers to a super (3, 2) and (3, 0) edge-antimagic total labelling of H_1, where these labels are assigned in same and in opposite order to the edges respectively.

Definition 6

For a path P_n, where n is even, G_2 is a non-planar graph of order $|V(G_1)|$ but non-isomorphic with following vertex and edge sets:

$$V(G_2)=\{x_i, y_i, z_i : 1\le i\le n\}\cup\{t_i, r_i : 1\le i\le \frac{n}{2}\}$$

$$E(G_2)=\{x_i\,y_i, y_i\,z_i, x_i\,z_i : 1\le i\le n\}\cup\{t_i\,x_{2i-1}, t_i\,x_{2i}, r_i\,z_{2i-1}, r_i\,z_{2i} : 1\le i\le \frac{n}{2}\}$$
$$\cup\{x_i\,x_{i+1}, y_i\,y_{i+1}, z_i\,z_{i+1} : 2\le i\le n-2, i\equiv 0 (\mathrm{mod}\, 2)\}$$
$$\cup\{t_i\,r_i : 1\le i\le \frac{n}{2}\}\cup\{x_i\,y_{i-1}, y_i\,z_{i-1} : 2\le i\le n, i\equiv 0 (\mathrm{mod}\, 2)\}$$

Theorem 3

The non-planar graph G_2 defined as above admits a super (3, 2) and (3, 0) edge- antimagic total labelling.

Proof. We define a labelling $\beta : V(G_2)\to\{1,2,...,4n\}$ as follows:

$$\beta(x_i)=\alpha(x_i) : 1\le i\le n;$$

$$\beta(y_i)=\begin{cases}\alpha(y_i) : 1\le i\le n-1; i\equiv 1(\mathrm{mod}\, 2);\\ \alpha(y_i) : 2\le i\le n; i\equiv 0(\mathrm{mod}\, 2);\end{cases}$$

$$\beta(z_i)=\alpha(z_i) : 1\le i\le n;$$

$$\beta(t_i)=\alpha(t_i) : 1\le i\le \frac{n}{2};$$

$$\beta(r_i)=\alpha(r_i) : 1\le i\le \frac{n}{2};$$

With same argument about edge- labels, this labelling β refers to a super (3, 2) and (3, 0) edge- antimagic total labelling of H_2, where these labels are assigned in same and in opposite order to the edges respectively.

In next theorems, we have found super edge-magic labelling of multi-cyclic family of graphs, which is a graph of order m that contains cycles of any lengths from 3 to m. In our case, this graph is termed as javelin graph JN_n, where n is the number of triangles C_3 on one side of JN_n. The graph JN_n is a graph of order $2(n+4)$ and size $4n+13$ with vertex and edge sets as:

For

$$n\equiv 0(\mathrm{mod}\, 2) :$$
$$V(JN_n)=\{x_i, y_i, z_i : 1\le i\le n+1\}\cup\{t_i : 1\le i\le 6\}.$$

$$E(JN_n) = \{t_1 x_i, t_2 y_i, t_2 x_i, t_2 y_i : 1 \le i \le \frac{n}{2}\} \cup \{t_i t_{i+1} : 1 \le i \le 5\}$$
$$\cup \{t_5 x_i, t_5 y_i, t_6 x_i, t_6 y_i : \frac{n+4}{2} \le i \le n+1\} \cup \{t_5 x_i, t_5 y_i, t_6 x_i, t_6 y_i : \frac{n+4}{2} \le i \le n+1\}$$
$$\cup \{x_{\frac{n+2}{2}} t_3, x_{\frac{n+2}{2}} t_4, y_{\frac{n+2}{2}} t_3, y_{\frac{n+2}{2}} t_4\} \cup \{x_{\frac{n+2}{2}} t_3, x_{\frac{n+2}{2}} t_4, y_{\frac{n+2}{2}} t_3, y_{\frac{n+2}{2}} t_4\}$$
$$\cup \{x_{\frac{n}{2}} x_{\frac{n+2}{2}}, x_{\frac{n+2}{2}} x_{\frac{n+4}{2}}, y_{\frac{n}{2}} y_{\frac{n+2}{2}},, y_{\frac{n+2}{2}} y_{\frac{n+4}{2}}\}$$

For

$n \equiv 1 (\text{mod } 2)$:

$V(JN_n) = \{x_i : 1 \le i \le n+2\} \cup \{y_i : 1 \le i \le n\} \cup \{t_i : 1 \le i \le 6\}$.

$$E(JN_n) = \{t_1 x_i, t_2 x_i : 1 \le i \le \frac{n+1}{2}\} \cup \{t_i t_{i+1} : 1 \le i \le 5\}$$
$$\cup \{t_5 x_i, t_6 x_i : \frac{n+5}{2} \le i \le n+2\} \cup \{t_1 y_i, t_2 y_i : 1 \le i \le \frac{n-1}{2}\}$$
$$\cup \{t_5 y_i, t_6 y_i : \frac{n+3}{2} \le i \le n\} \cup \{x_{\frac{n+3}{2}} t_3, x_{\frac{n+3}{2}} t_4, y_{\frac{n+1}{2}} t_3, y_{\frac{n+1}{2}} t_4\}$$
$$\cup \{x_{\frac{n+1}{2}} x_{\frac{n+3}{2}}, x_{\frac{n+3}{2}} x_{\frac{n+5}{2}}, y_{\frac{n-1}{2}} y_{\frac{n+1}{2}},, y_{\frac{n+1}{2}} y_{\frac{n+3}{2}}\}$$

Theorem 4

For even n, Javelin graph JN_n admits a super (3, 2) and (3, 0) edge- antimagic total labelling.

Proof. We define a labelling $f : V(JN_n) \to \{1,2,...,2(n+4)\}$ as follows:

$$f(x_i) = \begin{cases} 2i : 1 \le i \le \frac{n}{2}; \\ n+4 : i = \frac{n+2}{2}; \\ 2(i+2) : \frac{n+4}{2} \le i \le n+1; \end{cases}$$

$$f(y_i) = \begin{cases} 2i+1 : 1 \le i \le \frac{n}{2}; \\ n+5 : i = \frac{n+2}{2}; \\ 2i+5 : \frac{n+4}{2} \le i \le n+1; \end{cases}$$

$$f(t_i) = \begin{cases} 1 : i = 1; \\ n+2 : i = 2; \\ n+3 : i = 3; \\ n+6 : i = 4; \\ n+7 : i = 5; \\ 2(n+4) : i = 6; \end{cases}$$

The suitable edge- labels, this labeling *f* refers to a super (3, 2) and (3, 0) edge- antimagic total labeling of JN_n, definitely these labels are assigned in same and in opposite order to the edges respectively.

In next theorem, we exhibit super edge-magic labeling of JN_n for odd *n*.

Theorem 5

For even *n*, Javelin graph JN_n admits a super (3, 2) and (3, 0) edge- antimagic total labelling.

Proof. We define a labelling $g : V(JN_n) \to \{1,2,...,2(n+4)\}$ as follows:

$$g(x_i) = \begin{cases} 2i : 1 \le i \le \dfrac{n+1}{2}; \\ n+5 : i = \dfrac{n+3}{2}; \\ 2(i+2) : \dfrac{n+5}{2} \le i \le n+1; \\ 2n+7 : i = n+2; \end{cases}$$

$$g(y_i) = \begin{cases} 2i+1 : 1 \le i \le \dfrac{n-1}{2}; \\ n+4 : i = \dfrac{n+1}{2}; \\ 2i+5 : \dfrac{n+3}{2} \le i \le n; \end{cases}$$

$$g(t_i) = \begin{cases} 1 : i = 1; \\ n+2 : i = 2; \\ n+3 : i = 3; \\ n+6 : i = 4; \\ n+7 : i = 5; \\ 2(n+4) : i = 6; \end{cases}$$

The suitable edge- labels, this labeling g refers to a super (3, 2) and (3, 0) edge- antimagic total labeling of JN_n, definitely these labels are assigned in same and in opposite order to the edges respectively for this result as well.

CONCLUSIONS

We have presented, keeping in mind the importance of the calculation of super (a, d) edge- antimagic labelling due to its applications in computer science, crystallography and elsewhere, super (a, d) edge- antimagic labelling of three different families of non-planar graphs, for possible values of a and d. We have also proposed few open problems in this paper for mathematicians. In the end, we are proposing another open problem to work upon.

Open Problem 3

Can you determine a super edge- antimagic total labelling of the above discussed graphs for other possible values of a and d.

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